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Application of Integration by Parts for Fractional Calculus in Solving Two Types of Fractional Definite Integrals

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***Abstract*: In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we solve two types of fractional definite integrals of fractional trigonometric functions. The exact solutions of these two types of fractional definite integrals can be obtained by using integration by parts for fractional calculus. Moreover, we give some examples to illustrate our results. On the other hand, our results are generalizations of the classical calculus results.**

***Keywords*: Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, fractional definite integrals, fractional trigonometric functions, integration by parts for fractional calculus.**

I. INTRODUCTION

The calculus founded by Newton and Leibniz is a very important scientific achievement in the history of mathematics. Fractional calculus was first proposed by the famous mathematician Hospital in 1695. A question is about what is $\frac{d^{1/2}x}{dx^{1/2}}$? After 124 years, Lacroix gave the right answer to this question for the first time that $\frac{d^{1/2}x}{dx^{1/2}}=\frac{2}{\sqrt{\pi}}x^{1/2}$. However, for a long time, due to the lack of practical application, fractional calculus has not been widely used. With the development of science and technology, especially since the 20th century, the theory and application of fractional calculus began to be widely concerned. Fractional calculus has become a powerful tool to study fractional differential equations and fractional functions, and has been widely used in the research of physics, electrical engineering, viscoelasticity, control theory, biology, economics, and other fields [1-14].

However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [15-19]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study the following two α-fractional definite integrals of fractional trigonometric functions:

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\right],$$

and

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(cos_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\right].$$

Where $0 < \alpha \leq 1$, and m is any positive integer. Using integration by parts for fractional calculus, the exact solutions of these two types of fractional definite integrals can be obtained. In addition, some examples are provided to illustrate our results. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D^{\alpha}_{x}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I^{\alpha}_{x}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([21])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D^{\alpha}_{x}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D^{\alpha}_{x}\right)[c] = 0. \tag{4}$$

We introduce the definition of fractional analytic function below.

Definition 2.3 ([22])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([23])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) (x - x_0)^{n\alpha}. \quad (7)$$

Equivalently,

$$f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha)$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}. \quad (8)$$

Definition 2.5 ([24])**:** If $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha n}. \quad (9)$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha 2n}, \quad (10)$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (2n+1)}. \quad (11)$$

Definition 2.6 ([25])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha (-1)}$.

Theorem 2.7 (integration by parts for fractional calculus) ([26])**:** *Assume that* $0 < \alpha \leq 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha \right) \left[f_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^\alpha \right) [g_\alpha(x^\alpha)] \right] = \left[f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \right]_{x=a}^{x=b} - \left({}_aI_b^\alpha \right) \left[g_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^\alpha \right) [f_\alpha(x^\alpha)] \right]. \quad (12)$$

Definition 2.8 ([27])**:** The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS AND EXAMPLES

In this section, we use integration by parts for fractional calculus to solve two types of fractional definite integrals of fractional trigonometric functions. On the other hand, some examples are provided to illustrate our results. At first, we need a lemma.

Lemma 3.1: *If* $0 < \alpha \leq 1$, p, q *are real numbers,* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function, then*

$$\left({}_pI_q^\alpha \right) [f_\alpha(x^\alpha)] = \left({}_pI_q^\alpha \right) \left[f_\alpha \left(\frac{1}{\Gamma(\alpha+1)} p^\alpha + \frac{1}{\Gamma(\alpha+1)} q^\alpha - \frac{1}{\Gamma(\alpha+1)} x^\alpha \right) \right], \quad (13)$$

Proof $\left({}_pI_q^\alpha \right) \left[f_\alpha \left(\frac{1}{\Gamma(\alpha+1)} p^\alpha + \frac{1}{\Gamma(\alpha+1)} q^\alpha - \frac{1}{\Gamma(\alpha+1)} x^\alpha \right) \right]$

$$= -\left({}_pI_q^\alpha \right) \left[f_\alpha \left(\frac{1}{\Gamma(\alpha+1)} p^\alpha + \frac{1}{\Gamma(\alpha+1)} q^\alpha - \frac{1}{\Gamma(\alpha+1)} x^\alpha \right) \otimes_\alpha \left({}_0D_x^\alpha \right) \left[\frac{1}{\Gamma(\alpha+1)} p^\alpha + \frac{1}{\Gamma(\alpha+1)} q^\alpha - \frac{1}{\Gamma(\alpha+1)} x^\alpha \right] \right]$$

$$= -\left({}_qI_p^\alpha \right) [f_\alpha(x^\alpha)]$$

$$= \left({}_pI_q^\alpha \right) [f_\alpha(x^\alpha)].$$

Q.e.d.

Theorem 3.2: *Let* $0 < \alpha \leq 1$, *and* m *be any positive integer, then*

$$\left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\right] = \left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(cos_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\right]$$

$$= \begin{cases} \frac{(m-1)!!}{m!!}\cdot\frac{T_\alpha}{4} & if\ m\ is\ even \\ \frac{(m-1)!!}{m!!} & if\ m\ is\ odd \end{cases} . \quad (14)$$

Proof Let $D_m = \left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\right]$, then by integration by parts for fractional calculus,

$$D_m = \left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha (m-1)}\otimes_\alpha\left({}_0D^{\alpha}_{x}\right)[-cos_\alpha(x^\alpha)]\right]$$

$$= \left[-cos_\alpha(x^\alpha)\otimes_\alpha\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha (m-1)}\right]_0^{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}} + (m-1)\left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha (m-2)}\otimes_\alpha\left(cos_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}\right]$$

$$= (m-1)\left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha (m-2)}\otimes_\alpha\left[1-\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}\right]\right]$$

$$= (m-1)\ D_{m-2} - (m-1)\ D_m. \quad (15)$$

Thus,

$$D_m = \frac{m-1}{m}\ D_{m-2}. \quad (16)$$

It follows that

$$D_{2k} = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot D_0, \quad (17)$$

$$D_{2k+1} = \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdot\ldots\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot D_1. \quad (18)$$

And

$$D_0 = \left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)[1] = \frac{T_\alpha}{4}, \quad (19)$$

$$D_1 = \left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)[sin_\alpha(x^\alpha)] = 1. \quad (20)$$

Therefore,

$$D_{2k} = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{T_\alpha}{4}, \quad (21)$$

$$D_{2k+1} = \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdot\ldots\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}. \quad (22)$$

That is,

$$D_m = \begin{cases} \frac{(m-1)!!}{m!!} \cdot \frac{T_\alpha}{4} & \text{if } m \text{ is even} \\ \frac{(m-1)!!}{m!!} & \text{if } m \text{ is odd} \end{cases} . \quad (23)$$

On the other hand, by Lemma 3.1, we obtain

$$\left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\right] = \left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(cos_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\right].$$

Hence, the desired results hold. Q.e.d.

Example 3.3: Let $0 < \alpha \le 1$, then

$$\left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha 4}\right] = \frac{3}{32}\cdot T_\alpha. \quad (24)$$

And

$$\left({}_0I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[\left(cos_\alpha(x^\alpha)\right)^{\otimes_\alpha 7}\right] = \frac{48}{105}. \quad (25)$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we study two types of fractional definite integrals of fractional trigonometric functions. Using integration by parts for fractional calculus, we can obtain exact solutions of these two types of fractional definite integrals. On the other hand, our results are generalizations of the traditional calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and applied mathematics.

REFERENCES

[1] F. Mainardi, Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models, World Scientific, 2010.

[2] A. Carpinteri, F. Mainardi, (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, Wien, 1997.

[3] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[4] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[5] R. L. Magin, Fractional calculus in bioengineering, 13th International Carpathian Control Conference, 2012.

[6] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[7] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[8] M. Teodor, Atanacković, Stevan Pilipović, Bogoljub Stanković, Dušan Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, Inc., 2014.

[9] V. V. Kulish, J. L. Lage, Application of fractional calculus to fluid mechanics, Journal of Fluids Engineering, vol. 124, pp. 803-806, 2002.

[10] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics, vol.23, pp. 397-404. 2002.

[11] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, Fractional calculus and fractional processes with applications to financial economics, theory and application, Elsevier Science and Technology, 2016.

[12] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[13] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[14] F. Duarte and J. A. T. Machado, Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators, Nonlinear Dynamics, vol. 29, no. 1-4, pp. 315-342, 2002.

[15] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[16] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[17] S. Das, Functional Fractional Calculus, 2nd ed. Springer-Verlag, 2011.

[18] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[19] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, USA, 1993.

[20] C. -H. Yu, Using integration by parts for fractional calculus to solve some fractional integral problems, International Journal of Electrical and Electronics Research, vol. 11, no. 2, pp. 1-5, 2023.

[21] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[22] C. -H. Yu, Study on some properties of fractional analytic function, International Journal of Mechanical and Industrial Technology, vol. 10, no. 1, pp. 31-35, 2022.

[23] C. -H. Yu, Exact solutions of some fractional power series, International Journal of Engineering Research and Reviews, vol. 11, no. 1, pp. 36-40, 2023.

[24] C. -H. Yu, Fractional differential problem of some fractional trigonometric functions, International Journal of Interdisciplinary Research and Innovations, vol. 10, no. 4, pp. 48-53, 2022.

[25] C. -H. Yu, Infinite series expressions for the values of some fractional analytic functions, International Journal of Interdisciplinary Research and Innovations, vol. 11, no. 1, pp. 80-85, 2023.

[26] C. -H. Yu, Using integration by parts for fractional calculus to solve some fractional integral problems, International Journal of Electrical and Electronics Research, vol. 11, no. 2, pp. 1-5, 2023.

[27] C. -H. Yu, Study of two fractional integrals, International Journal of Novel Research in Physics Chemistry & Mathematics, vol. 10, no. 2, pp. 1-6, 2023.